

Technical Note

Extending the Duhamel theorem to dual phase applications

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Abstract

The Duhamel theorem is a useful classical result that allows finding the solution to a single phase thermal diffusion problem subject to time dependent heat sources and time dependent boundary conditions in terms of known solutions to the equivalent problem when the heat sources and boundary conditions are independent of time. The present paper presents the proof to the Duhamel theorem for dual phase thermal diffusion applications.

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1. Introduction

The Duhamel theorem ([1–4]) is a very useful classical result that allows one to find the solution to a single phase thermal diffusion problem subject to time dependent heat sources and time dependent boundary conditions in terms of known solutions to the equivalent problem when the heat sources and boundary conditions are independent of time. Kast [5] derived recently an extension of the Duhamel theorem to stochastic boundary conditions. The formulation of the Duhamel theorem does not allow for its direct application to dual-phase conditions. The latter include in particular porous media applications subject to lack of local thermal equilibrium (LaLotheq) such as Minkowycz, Haji-Sheikh and Vafai [6], Nield [7,8], Nield and Bejan [9] and Vadasz [10–13] as well as solid suspensions in fluids as applied, for example, for nanofluids by Vadasz [14]. Additional applications include any combination of two phases where the heat transfer in each phase as well as over

the interface separating the two phases occurs by thermal diffusion. In particular modern experimental techniques such as the transient hot wire method for the evaluation of the thermal conductivity are not directly applicable to dual-phase applications. The rendering of such methods to dual-phase systems requires the development of the theoretical and analytical basis for such applications. The objective of the present paper is to provide the extension of the Duhamel theorem for such dual phase applications of thermal diffusion including its rigorous proof.

2. Problem formulation and local thermal equilibrium

Let us consider the heat conduction in any dual phase system. While all the following derivations and conclusions apply generally as stated above to any dual phase system we adopt here the notation that is particularly useful for porous media applications, i.e. the subscripts s and f represent the solid and fluid phases, respectively. The equivalent application to any other dual phase system is obtained by allocating the subscripts s and f to represent phases 1 and 2, respectively. Heat conduction in porous media subject to lack of local thermal equilibrium (LaLotheq) is governed at the macro-level by the following equations that

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Nomenclature

Latin symbols

h	integral heat transfer coefficient for the heat conduction at the interface between two phases
k_s	effective thermal conductivity of the solid phase, equals $(1 - \varepsilon)\tilde{k}_s$
\tilde{k}_s	thermal conductivity of the solid phase
k_f	effective thermal conductivity of the fluid phase, equals $\varepsilon\tilde{k}_f$
\tilde{k}_f	thermal conductivity of the fluid phase
t	time
T	temperature
\mathbf{r}	position vector, in Cartesian coordinates equals $(x\hat{\mathbf{e}}_x, y\hat{\mathbf{e}}_y, z\hat{\mathbf{e}}_z)$

Greek symbols

α_s	solid phase effective thermal diffusivity, equals k_s/γ_s (dimensional)
α_f	fluid phase effective thermal diffusivity, equals k_f/γ_f

γ_s	solid phase effective heat capacity, equals $(1 - \varepsilon)\rho_s c_s$
γ_f	fluid phase effective heat capacity, equals $\varepsilon\rho_f c_{p,f}$
ε	porosity
ρ_s	solid phase density
ρ_f	fluid phase density

Subscripts

s	related to the solid phase
f	related to the fluid phase

represent averages over each phase within an REV (representative elementary volume)

$$\gamma_s \frac{\partial T_s}{\partial t} = k_s \nabla^2 T_s - h(T_s - T_f) + g_s(\mathbf{r}, t) \quad (1)$$

$$\gamma_f \frac{\partial T_f}{\partial t} = k_f \nabla^2 T_f + h(T_s - T_f) + g_f(\mathbf{r}, t) \quad (2)$$

where $Q_{sf} = h(T_s - T_f)$ represents the rate of heat generation in the fluid phase within the REV due to the heat transferred over the fluid–solid interface, and where $\gamma_s = (1 - \varepsilon)\rho_s c_s$ and $\gamma_f = \varepsilon\rho_f c_{p,f}$ are the solid phase and fluid phase effective heat capacities, respectively, ε is the porosity, $k_s = (1 - \varepsilon)\tilde{k}_s$ and $k_f = \varepsilon\tilde{k}_f$ are the effective thermal conductivities of the solid and fluid phases, respectively. The coefficient $h > 0$, carrying units of $\text{W m}^{-3} \text{K}^{-1}$, is a macro-level integral heat transfer coefficient for the heat conduction at the fluid–solid interface (averaged over the REV). The heat sources/sinks terms in (1) and (2) are general and allow for different and time dependent heat source functions in each phase.

Eqs. (1) and (2) are linearly coupled and represent the traditional form of expressing the process of heat conduction in porous media subject to LaLotheq (Nield and Bejan [9], Nield [7]). When the value of the interface heat transfer coefficient vanishes, $h = 0$, (physically representing an interface that is an ideal insulator, e.g. the solid–fluid interface is coated with a highly insulating material) Eqs. (1) and (2) un-couple and the solution for the temperature of each phase is independent of the other phase, the phase having the highest thermal diffusivity producing a temperature that equilibrates faster to its steady state value. Very large values of h on the other hand lead to local thermal equilibrium (Lotheq) as observed by dividing Eqs. (1)

and (2) by h and looking for the limit as $h \rightarrow \infty$ that produces (at least at the leading order) $T_s = T_f$. The latter occurs because despite the fact that one phase (the slow one) diffuses heat at a slower pace a perfect compensation occurs due to the interface heat transfer, i.e. the change in temperature in the faster phase is immediately followed by an identical temperature change in the slower phase via the heat transferred through the interface without any resistance because $h \rightarrow \infty$. Similar results may be obtained with a finite interface heat transfer coefficient, h , if the thermal diffusivities of both phases are identical, i.e. $\alpha_s = (k_s/\gamma_s) = (k_f/\gamma_f) = \alpha_f$. Then, both phases will diffuse heat at the same pace leading naturally to $T_s = T_f = T$ and a vanishing heat transfer over the interface $h(T_s - T_f) = 0$ irrespective of the value of h . Eqs. (1) and (2) are subject to the following general time dependent boundary conditions

$$k_s \frac{\partial T_s}{\partial n_i} + h_{si} T_s = f_{si}(\mathbf{r}_b, t) \quad \text{on the boundary } S_i \text{ for } t > 0 \quad (3)$$

$$k_f \frac{\partial T_f}{\partial n_i} + h_{fi} T_f = f_{fi}(\mathbf{r}_b, t) \quad \text{on the boundary } S_i \text{ for } t > 0 \quad (4)$$

where \mathbf{r}_b represents the equation for the geometry of the boundary surface $S_i \hat{\mathbf{e}}_i$, and where $\hat{\mathbf{e}}_i$ is a unit vector normal to the boundary surface, facing outwards. Note that the constant parameters h_{si} and h_{fi} are fundamentally distinct than the interface heat transfer coefficient h , despite the similar notation. By setting $k_s = 0$, $h_{si} = 1$, $k_f = 0$, $h_{fi} = 1$ in Eqs. (3) and (4) we obtain Dirichlet type of boundary

conditions, and by setting $h_{si} = 0$, $h_{fi} = 0$ we obtain Neumann type of boundary conditions. The initial conditions are

$$T_s(\mathbf{r}, 0) = F_s(\mathbf{r}) \text{ in } R \text{ for } t = 0 \tag{5}$$

$$T_f(\mathbf{r}, 0) = F_f(\mathbf{r}) \text{ in } R \text{ for } t = 0 \tag{6}$$

where \mathbf{r} represents the position vector, which in Cartesian coordinates is defined in the form $\mathbf{r} = x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y + z\hat{\mathbf{e}}_z$, and where $\hat{\mathbf{e}}_x$, $\hat{\mathbf{e}}_y$ and $\hat{\mathbf{e}}_z$ are unit vectors in the x , y and z directions, respectively.

When local thermal equilibrium (Lotheq) applies $T_s = T_f = T$ and Eqs. (1)–(4) may be added to yield

$$\gamma_e \frac{\partial T(\mathbf{r}, t)}{\partial t} = k_e \nabla^2 T(\mathbf{r}, t) + g_e(\mathbf{r}, t) \tag{7}$$

where $\gamma_e = \gamma_s + \gamma_f$, $k_e = k_s + k_f$ and $g_e(\mathbf{r}, t) = g_s(\mathbf{r}, t) + g_f(\mathbf{r}, t)$, subject to the boundary conditions

$$k_e \frac{\partial T}{\partial n_i} + h_{ei} T = f_{ei}(\mathbf{r}_b, t) \text{ on the boundary } S_i \text{ for } t > 0 \tag{8}$$

where $h_{ei} = h_{si} + h_{fi}$ and $f_{ei}(\mathbf{r}_b, t) = f_{si}(\mathbf{r}_b, t) + f_{fi}(\mathbf{r}_b, t)$. The initial conditions must be identical for both phases for Lotheq conditions to apply, i.e.

$$T(\mathbf{r}, 0) = F(\mathbf{r}) \text{ in } R \text{ for } t = 0 \tag{9}$$

Then, when local thermal equilibrium (Lotheq) conditions apply the Duhamel theorem for single phase thermal diffusion is applicable in the form

$$T(\mathbf{r}, t) = \frac{\partial}{\partial t} \int_0^t \varphi(\mathbf{r}, t - \tau, \tau) d\tau \tag{10}$$

stating that the local thermal equilibrium (Lotheq) temperature solution $T(\mathbf{r}, t)$, to problem (7)–(9) that includes the time dependent source term $g_e(\mathbf{r}, t)$ and is subject to the time dependent boundary conditions (8), and initial conditions (9), is obtained from the known local thermal equilibrium (Lotheq) temperature solution $\varphi(\mathbf{r}, t, \tau)$, to the problem

$$\gamma_e \frac{\partial \varphi(\mathbf{r}, t, \tau)}{\partial t} = k_e \nabla^2 \varphi(\mathbf{r}, t, \tau) + g_e(\mathbf{r}, \tau) \tag{11}$$

subject to the boundary and initial conditions

$$k_e \frac{\partial \varphi}{\partial n_i} + h_{ei} \varphi = f_{ei}(\mathbf{r}_b, \tau) \text{ on the boundary } S_i \text{ for } t > 0 \tag{12}$$

$$\varphi(\mathbf{r}, 0) = F(\mathbf{r}) \text{ in } R \text{ for } t = 0 \tag{13}$$

where $g_e(\mathbf{r}, \tau)$ and $f_{ei}(\mathbf{r}_b, \tau)$ are independent of time, τ being a fixed parameter.

Eq. (10) represents the Duhamel theorem (Duhamel [1], Carslaw and Jaeger [3], Özisik [4]) for single phase thermal diffusion and is applicable to dual phase applications if local thermal equilibrium exists. Bartels and Churchill [2] provided a general proof of (10) to single phase applica-

tions including extensions to other systems and a more general formulation of (7) and (11) in the form

$$\gamma_e \frac{\partial T(\mathbf{r}, t)}{\partial t} = k_e \nabla^2 T(\mathbf{r}, t) + bT(\mathbf{r}, t) + g_e(\mathbf{r}, t)$$

$$\gamma_e \frac{\partial \varphi(\mathbf{r}, t, \tau)}{\partial t} = k_e \nabla^2 \varphi(\mathbf{r}, t, \tau) + b\varphi(\mathbf{r}, t, \tau) + g_e(\mathbf{r}, \tau)$$

for any positive, negative or zero value of $b = \text{constant}$. One may attempt to apply this extended formulation to dual phase LaLotheq systems by choosing $b = -h$, and $g_s(\mathbf{r}, t) = hT_f(\mathbf{r}, t)$ in the equation for the solid phase and $g_f(\mathbf{r}, t) = hT_s(\mathbf{r}, t)$ in the equation for the fluid phase. However, the latter does not resolve the problem because in the application of Eq. (10) the source term time dependence is frozen to a fixed value τ causing for example for the solid phase $g_s(\mathbf{r}, \tau) = hT_f(\mathbf{r}, \tau)$ while we need the time dependence of $T_f(\mathbf{r}, \tau)$ to solve the coupled equations. Therefore, Eq. (10) cannot be used directly to dual phase systems subject to LaLotheq. In addition, when dual phase systems are lacking local thermal equilibrium (LaLotheq) there is no proof nor derived relationship equivalent to (10). The next sections focus on deriving the equivalent Duhamel formulation for dual phase systems subject to LaLotheq and to providing such a proof.

3. The Laplace transform of the convolution

As the method of proof for the dual phase system subject to lack of local thermal equilibrium (LaLotheq) involves the application of Laplace transforms in general and Laplace transforms of the convolution in particular [2] it becomes appealing to introduce the latter in this section in order to provide relationships that are useful in the following section.

The generalized convolution $\varphi^*(\mathbf{r}, t)$ of the function $\varphi(\mathbf{r}, t, \tau)$ is defined by [2]

$$\varphi^*(\mathbf{r}, t) = \int_0^t \varphi(\mathbf{r}, t - \tau, \tau) d\tau \tag{14}$$

From (14) it is obvious to observe that $\varphi^*(\mathbf{r}, 0) = 0$. The variable \mathbf{r} represents the position vector, which in Cartesian coordinates is defined in the form $\mathbf{r} = x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y + z\hat{\mathbf{e}}_z$, where $\hat{\mathbf{e}}_x$, $\hat{\mathbf{e}}_y$ and $\hat{\mathbf{e}}_z$ are unit vectors in the x , y and z directions, and is not relevant to this first part of the derivations, but becomes relevant later. In case $\varphi(\mathbf{r}, t, \tau) = \varphi_1(\mathbf{r}, t)\varphi_2(\mathbf{r}, \tau)$ the function $\varphi^*(\mathbf{r}, t)$ is the ordinary convolution $\varphi_1(\mathbf{r}, t) * \varphi_2(\mathbf{r}, \tau)$ of the two functions $\varphi_1(\mathbf{r}, t)$ and $\varphi_2(\mathbf{r}, \tau)$.

Let $\bar{\varphi}^*(\mathbf{r}, s) \equiv \mathcal{L}[\varphi^*(\mathbf{r}, t)]$ denote the Laplace transform of $\varphi^*(\mathbf{r}, t)$ with respect to t , defined by

$$\bar{\varphi}^*(\mathbf{r}, s) \equiv \mathcal{L}[\varphi^*(\mathbf{r}, t)] = \int_0^\infty e^{-st} \varphi^*(\mathbf{r}, t) dt \tag{15}$$

and let $\bar{\bar{\varphi}}(\mathbf{r}, s)$ denote the iterated transform of $\varphi(\mathbf{r}, t, \tau)$ first with respect with t and then with respect with τ , in the form [2]

$$\begin{aligned} \bar{\bar{\varphi}}(\mathbf{r}, s) &\equiv \mathcal{L}[\bar{\varphi}(\mathbf{r}, \tau)] = \int_{\tau=0}^{\infty} e^{-s\tau} \bar{\varphi}(\mathbf{r}, \tau) d\tau \\ &= \int_{\tau=0}^{\infty} e^{-s\tau} d\tau \int_{t=0}^{\infty} e^{-st} \varphi(\mathbf{r}, t, \tau) dt \end{aligned} \tag{16}$$

Bartels and Churchill [2] introduced and proved the following equation relating $\bar{\bar{\varphi}}(\mathbf{r}, s)$ to $\bar{\varphi}^*(\mathbf{r}, s)$

$$\bar{\varphi}^*(\mathbf{r}, s) = \bar{\bar{\varphi}}(\mathbf{r}, s) \tag{17}$$

We will not repeat this proof here and the reader is referred to Bartels and Churchill [2] for the details of the proof.

4. Duhamel theorem for dual phase systems (LaLotheq)

Consider the dual phase thermal diffusion initial-boundary value problem subject to LaLotheq

$$\begin{aligned} \gamma_s \frac{\partial T_s(\mathbf{r}, t)}{\partial t} &= k_s \nabla^2 T_s(\mathbf{r}, t) - h[T_s(\mathbf{r}, t) - T_f(\mathbf{r}, t)] \\ &+ g_s(\mathbf{r}, t) \text{ in } R \text{ for } t \geq 0 \end{aligned} \tag{18}$$

$$\begin{aligned} \gamma_f \frac{\partial T_f(\mathbf{r}, t)}{\partial t} &= k_f \nabla^2 T_f(\mathbf{r}, t) + h[T_s(\mathbf{r}, t) - T_f(\mathbf{r}, t)] \\ &+ g_f(\mathbf{r}, t) \text{ in } R \text{ for } t \geq 0 \end{aligned} \tag{19}$$

where the definitions are identical to the ones introduced following Eqs. (1) and (2). The time dependent boundary conditions are

$$k_s \frac{\partial T_s}{\partial n_i} + h_{si} T_s = f_{si}(\mathbf{r}_b, t) \text{ on the boundary } S_i \text{ for } t > 0 \tag{20}$$

$$k_f \frac{\partial T_f}{\partial n_i} + h_{fi} T_f = f_{fi}(\mathbf{r}_b, t) \text{ on the boundary } S_i \text{ for } t > 0 \tag{21}$$

and the initial conditions are

$$T_s(\mathbf{r}, 0) = F_s(\mathbf{r}) \text{ in } R \text{ for } t = 0 \tag{22}$$

$$T_f(\mathbf{r}, 0) = F_f(\mathbf{r}) \text{ in } R \text{ for } t = 0 \tag{23}$$

Applying now the Laplace transform on Eqs. (18)–(21) by using the definition of the Laplace transform and the Laplace transform of a time derivative yields

$$\begin{aligned} \gamma_s s \bar{T}_s(\mathbf{r}, s) - \gamma_s F_s(\mathbf{r}) &= k_s \nabla^2 \bar{T}_s(\mathbf{r}, s) - h[\bar{T}_s(\mathbf{r}, s) \\ &- \bar{T}_f(\mathbf{r}, s)] + \bar{g}_s(\mathbf{r}, s) \end{aligned} \tag{24}$$

$$\begin{aligned} \gamma_f s \bar{T}_f(\mathbf{r}, s) - \gamma_f F_f(\mathbf{r}) &= k_f \nabla^2 \bar{T}_f(\mathbf{r}, s) + h[\bar{T}_s(\mathbf{r}, s) \\ &- \bar{T}_f(\mathbf{r}, s)] + \bar{g}_f(\mathbf{r}, s) \end{aligned} \tag{25}$$

$$k_s \frac{\partial \bar{T}_s}{\partial n_i} + h_{si} \bar{T}_s = \bar{f}_{si}(\mathbf{r}_b, s) \tag{26}$$

$$k_f \frac{\partial \bar{T}_f}{\partial n_i} + h_{fi} \bar{T}_f = \bar{f}_{fi}(\mathbf{r}_b, s) \tag{27}$$

Now let $\varphi_s(\mathbf{r}, t, \tau)$ and $\varphi_f(\mathbf{r}, t, \tau)$ be the solution to the following problem

$$\begin{aligned} \gamma_s \frac{\partial \varphi_s(\mathbf{r}, t, \tau)}{\partial t} &= k_s \nabla^2 \varphi_s(\mathbf{r}, t, \tau) - h[\varphi_s(\mathbf{r}, t, \tau) \\ &- \varphi_f(\mathbf{r}, t, \tau)] + g_s(\mathbf{r}, \tau) \text{ in } R \text{ for } t \geq 0 \end{aligned} \tag{28}$$

$$\begin{aligned} \gamma_f \frac{\partial \varphi_f(\mathbf{r}, t, \tau)}{\partial t} &= k_f \nabla^2 \varphi_f(\mathbf{r}, t, \tau) + h[\varphi_s(\mathbf{r}, t, \tau) \\ &- \varphi_f(\mathbf{r}, t, \tau)] + g_f(\mathbf{r}, \tau) \text{ in } R \text{ for } t \geq 0 \end{aligned} \tag{29}$$

subject to the boundary conditions

$$k_s \frac{\partial \varphi_s}{\partial n_i} + h_{si} \varphi_s = f_{si}(\mathbf{r}_b, \tau) \text{ on the boundary } S_i \text{ for } t > 0 \tag{30}$$

$$k_f \frac{\partial \varphi_f}{\partial n_i} + h_{fi} \varphi_f = f_{fi}(\mathbf{r}_b, \tau) \text{ on the boundary } S_i \text{ for } t > 0 \tag{31}$$

and the initial conditions

$$\varphi_s(\mathbf{r}, 0) = F_s(\mathbf{r}) \text{ in } R \text{ for } t = 0 \tag{32}$$

$$\varphi_f(\mathbf{r}, 0) = F_f(\mathbf{r}) \text{ in } R \text{ for } t = 0 \tag{33}$$

where $g_s(\mathbf{r}, \tau)$, $g_f(\mathbf{r}, \tau)$ and $f_{si}(\mathbf{r}_b, \tau)$, $f_{fi}(\mathbf{r}_b, \tau)$ are independent of time, τ being a fixed parameter.

Then the Laplace transform of Eqs. (28)–(31) with respect to the variable t , obtained by using the definition of the Laplace transform, the Laplace transform of a time derivative and the Laplace transform of a constant, yields

$$\begin{aligned} \gamma_s s \bar{\bar{\varphi}}_s(\mathbf{r}, s, \tau) - \gamma_s F_s(\mathbf{r}) &= k_s \nabla^2 \bar{\bar{\varphi}}_s(\mathbf{r}, s, \tau) - h[\bar{\bar{\varphi}}_s(\mathbf{r}, s, \tau) \\ &- \bar{\bar{\varphi}}_f(\mathbf{r}, s, \tau)] + \frac{1}{s} g_s(\mathbf{r}, \tau) \end{aligned} \tag{34}$$

$$\begin{aligned} \gamma_f s \bar{\bar{\varphi}}_f(\mathbf{r}, s, \tau) - \gamma_f F_f(\mathbf{r}) &= k_f \nabla^2 \bar{\bar{\varphi}}_f(\mathbf{r}, s, \tau) + h[\bar{\bar{\varphi}}_s(\mathbf{r}, s, \tau) \\ &- \bar{\bar{\varphi}}_f(\mathbf{r}, s, \tau)] + \frac{1}{s} g_f(\mathbf{r}, \tau) \end{aligned} \tag{35}$$

$$k_s \frac{\partial \bar{\bar{\varphi}}_s}{\partial n_i} + h_{si} \bar{\bar{\varphi}}_s = \frac{1}{s} f_{si}(\mathbf{r}_b, \tau) \tag{36}$$

$$k_f \frac{\partial \bar{\bar{\varphi}}_f}{\partial n_i} + h_{fi} \bar{\bar{\varphi}}_f = \frac{1}{s} f_{fi}(\mathbf{r}_b, \tau) \tag{37}$$

Applying now on Eqs. (34)–(37) the Laplace transform with respect to τ , defined in the form $\bar{p}(\mathbf{r}, s) \equiv \mathcal{L}[p(\mathbf{r}, \tau)] = \int_0^\infty e^{-s\tau} p(\mathbf{r}, \tau) d\tau$, and the definition of the iterated transform $\bar{\bar{\varphi}}(\mathbf{r}, s)$ of a function $\varphi(\mathbf{r}, t, \tau)$ from Eq. (16), yields

$$\begin{aligned} \gamma_s s \bar{\bar{\bar{\varphi}}}_s(\mathbf{r}, s) - \frac{1}{s} \gamma_s F_s(\mathbf{r}) &= k_s \nabla^2 \bar{\bar{\bar{\varphi}}}_s(\mathbf{r}, s) - h[\bar{\bar{\bar{\varphi}}}_s(\mathbf{r}, s) \\ &- \bar{\bar{\bar{\varphi}}}_f(\mathbf{r}, s)] + \frac{1}{s} \bar{g}_s(\mathbf{r}, s) \end{aligned} \tag{38}$$

$$\begin{aligned} \gamma_f s \bar{\bar{\bar{\varphi}}}_f(\mathbf{r}, s) - \frac{1}{s} \gamma_f F_f(\mathbf{r}) &= k_f \nabla^2 \bar{\bar{\bar{\varphi}}}_f(\mathbf{r}, s) + h[\bar{\bar{\bar{\varphi}}}_s(\mathbf{r}, s) \\ &- \bar{\bar{\bar{\varphi}}}_f(\mathbf{r}, s)] + \frac{1}{s} \bar{g}_f(\mathbf{r}, s) \end{aligned} \tag{39}$$

$$k_s \frac{\partial \bar{\bar{\bar{\varphi}}}_s}{\partial n_i} + h_{si} \bar{\bar{\bar{\varphi}}}_s = \frac{1}{s} \bar{f}_{si}(\mathbf{r}_b, s) \tag{40}$$

$$k_f \frac{\partial \bar{\bar{\bar{\varphi}}}_f}{\partial n_i} + h_{fi} \bar{\bar{\bar{\varphi}}}_f = \frac{1}{s} \bar{f}_{fi}(\mathbf{r}_b, s) \tag{41}$$

Multiplying Eqs. (38)–(41) by s produces

$$\gamma_s s^2 \bar{\bar{\varphi}}_s(\mathbf{r}, s) - \gamma_s F_s(\mathbf{r}) = k_s \nabla^2 s \bar{\bar{\varphi}}_s(\mathbf{r}, s) - h[s \bar{\bar{\varphi}}_s(\mathbf{r}, s) - s \bar{\bar{\varphi}}_f(\mathbf{r}, s)] + \bar{g}_s(\mathbf{r}, s) \quad (42)$$

$$\gamma_f s^2 \bar{\bar{\varphi}}_f(\mathbf{r}, s) - \gamma_f F_f(\mathbf{r}) = k_f \nabla^2 s \bar{\bar{\varphi}}_f(\mathbf{r}, s) + h[s \bar{\bar{\varphi}}_s(\mathbf{r}, s) - s \bar{\bar{\varphi}}_f(\mathbf{r}, s)] + \bar{g}_f(\mathbf{r}, s) \quad (43)$$

$$k_s \frac{\partial (s \bar{\bar{\varphi}}_s)}{\partial n_i} + h_{si}(s \bar{\bar{\varphi}}_s) = \bar{f}_{si}(\mathbf{r}_b, s) \quad (44)$$

$$k_f \frac{\partial (s \bar{\bar{\varphi}}_f)}{\partial n_i} + h_{fi}(s \bar{\bar{\varphi}}_f) = \bar{f}_{fi}(\mathbf{r}_b, s) \quad (45)$$

Comparing Eqs. (42)–(45) with Eqs. (24)–(27) leads to the conclusion that they are identical if and only if

$$\bar{T}_s(\mathbf{r}, s) = s \bar{\bar{\varphi}}_s(\mathbf{r}, s) \quad \text{and} \quad \bar{T}_f(\mathbf{r}, s) = s \bar{\bar{\varphi}}_f(\mathbf{r}, s) \quad (46)$$

However, according to Eq. (17) we have

$$\bar{\varphi}_s^*(\mathbf{r}, s) = \bar{\bar{\varphi}}_s(\mathbf{r}, s) \quad \text{and} \quad \bar{\varphi}_f^*(\mathbf{r}, s) = \bar{\bar{\varphi}}_f(\mathbf{r}, s) \quad (47)$$

Combining (46) and (47) leads to

$$\bar{T}_s(\mathbf{r}, s) = s \bar{\varphi}_s^*(\mathbf{r}, s) \quad \text{and} \quad \bar{T}_f(\mathbf{r}, s) = s \bar{\varphi}_f^*(\mathbf{r}, s) \quad (48)$$

But in addition we also have the following relationships for the Laplace transform of a time derivative

$$\mathcal{L} \left[\frac{\partial \varphi_s^*(\mathbf{r}, t)}{\partial t} \right] = s \bar{\varphi}_s^*(\mathbf{r}, s) - \underbrace{\varphi_s^*(\mathbf{r}, 0)}_{=0 \text{ by Eq. (14)}} = s \bar{\varphi}_s^*(\mathbf{r}, s) \quad (49)$$

$$\mathcal{L} \left[\frac{\partial \varphi_f^*(\mathbf{r}, t)}{\partial t} \right] = s \bar{\varphi}_f^*(\mathbf{r}, s) - \underbrace{\varphi_f^*(\mathbf{r}, 0)}_{=0 \text{ by Eq. (14)}} = s \bar{\varphi}_f^*(\mathbf{r}, s) \quad (50)$$

Combining (48) with (49) and (50) yields

$$\bar{T}_s(\mathbf{r}, s) \equiv \mathcal{L}[T_s(\mathbf{r}, t)] = \mathcal{L} \left[\frac{\partial \varphi_s^*(\mathbf{r}, t)}{\partial t} \right] \quad (51)$$

$$\bar{T}_f(\mathbf{r}, s) \equiv \mathcal{L}[T_f(\mathbf{r}, t)] = \mathcal{L} \left[\frac{\partial \varphi_f^*(\mathbf{r}, t)}{\partial t} \right] \quad (52)$$

Taking now the inverse Laplace transform of (51) and (52)

$$T_s(\mathbf{r}, t) \equiv \mathcal{L}^{-1} \mathcal{L}[T_s(\mathbf{r}, t)] = \mathcal{L}^{-1} \mathcal{L} \left[\frac{\partial \varphi_s^*(\mathbf{r}, t)}{\partial t} \right] = \frac{\partial \varphi_s^*(\mathbf{r}, t)}{\partial t} \quad (53)$$

$$T_f(\mathbf{r}, t) \equiv \mathcal{L}^{-1} \mathcal{L}[T_f(\mathbf{r}, t)] = \mathcal{L}^{-1} \mathcal{L} \left[\frac{\partial \varphi_f^*(\mathbf{r}, t)}{\partial t} \right] = \frac{\partial \varphi_f^*(\mathbf{r}, t)}{\partial t} \quad (54)$$

Substituting the definition of the convolution (14) into (53) and (54) produces the solutions for $T_s(\mathbf{r}, t)$ and $T_f(\mathbf{r}, t)$ in terms of the known solutions $\varphi_s(\mathbf{r}, t, \tau)$ and $\varphi_f(\mathbf{r}, t, \tau)$ to problem (28)–(33) in the form

$$T_s(\mathbf{r}, t) = \frac{\partial}{\partial t} \int_0^t \varphi_s(\mathbf{r}, t - \tau, \tau) d\tau \quad (55)$$

$$T_f(\mathbf{r}, t) = \frac{\partial}{\partial t} \int_0^t \varphi_f(\mathbf{r}, t - \tau, \tau) d\tau \quad (56)$$

Eqs. (55) and (56) represent the Duhamel theorem for dual phase systems. An alternative form of (55) and (56) can be

obtained by using the Leibnitz rule for derivatives of definite integrals leading to

$$T_s(\mathbf{r}, t) = \int_0^t \frac{\partial \varphi_s(\mathbf{r}, t - \tau, \tau)}{\partial t} d\tau + F_s(\mathbf{r}) \quad (57)$$

$$T_f(\mathbf{r}, t) = \int_0^t \frac{\partial \varphi_f(\mathbf{r}, t - \tau, \tau)}{\partial t} d\tau + F_f(\mathbf{r}) \quad (58)$$

5. Conclusions

The formulation of the Duhamel theorem applicable to dual phase systems, such as porous media and solid suspensions in fluids was derived and a proof for its applicability was provided along with the derivations. The Duhamel theorem for dual phase systems subject to time dependent heat sources and time dependent boundary conditions are expressed in terms of known solutions of the equivalent problem when the heat sources and boundary conditions are independent of time, by Eqs. (55) and (56) or alternatively (57) and (58), and is linked to the solution of problem (28)–(33).

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